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THE "RESTRICTED" FORMULATION OF THE PROBLEM OF THE MOTION OF A HEAVY RIGID BODY†

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The problem of the motion of a heavy rigid body is considered in the so-called "restricted" formulation, which is obtained on the assumption that two dimensions of the body, which we will call its "width" and "thickness", are considerably less than the third dimension, the "length" of the body. They dynamics of the limiting objects which arise are investigated; in particular, the question of the existence and stability of the steady motions, the separation of motions, and the integration and integrability of the equations of motion is considered. © 2005 Elsevier Ltd. All rights reserved.

The problem of restricted formulations of problems of rigid body dynamics was formulated in [1] (see also [2]) when investigating the main properties of the limit problems of the dynamics of a heavy rigid body with a fixed point and the dynamics of a rigid body in an ideal incompressible fluid, at rest at infinity. Below, developing an idea put forward in [3], we suggest a method of introducing a parameter, characterizing the dimensions of the body, which differs somewhat from the method used previously in [1] and enables a wider class of problems with more abundant dynamic properties to be investigated.

1. THE GENERAL EQUATIONS OF MOTION

Consider the motion of a heavy rigid body about a fixed point. For simplicity we will assume that the body consists of a certain number of point masses A_i with masses m_i, $i \in \mathcal{I}$. Suppose $Ox_1x_2x_3$ is a body-fixed system of coordinates, the origin of which coincides with the fixed point O, and whose axes are directed along the principal axes of inertia about the point O. The position of the points A_i is specified by the vectors $\overrightarrow{OA}_i$, the projections of which on to the axes of this system of coordinates have the form $\mathbf{r}_i = (r_{1i}, r_{2i}, r_{3i})$.

If g is the acceleration due to gravity, $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3) \in R^3(\boldsymbol{\omega})$ is the angular velocity vector and $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3) \in R^3(\boldsymbol{\gamma})$ is the unit vector, directed along the ascending vertical, the equations of motion have the form

$$(\Lambda_2 + \Lambda_3)\dot{\omega}_1 = (\Lambda_3 - \Lambda_2)\omega_2\omega_3 + \gamma_2 M_3 - \gamma_3 M_2 \ (1, 2, 3) \tag{1.1}$$

$$\dot{\gamma}_1 = \gamma_2\omega_3 - \gamma_3\omega_2 \ (1, 2, 3) \tag{1.2}$$

$$\Lambda_j = \sum_{i \in \mathcal{I}} m_i r_{ji}^2, \quad M_j = g\sum_{i \in \mathcal{I}} m_i r_{ji}, \quad M = \sum_{i \in \mathcal{I}} m_i \tag{1.3}$$

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The system of Euler–Poisson equations (1.1), (1.2), as is well known, in addition to the energy integral

$$\mathcal{J}_0 = \frac{1}{2}\sum_{(1,2,3)}(\Lambda_2+\Lambda_3)\omega_1^2 + \sum_{(1,2,3)} M_1\gamma_1 = h \tag{1.4}$$

allows of the area integral

$$\mathcal{J}_1 = \sum_{(1,2,3)}(\Lambda_2+\Lambda_3)\omega_1\gamma_1 = p_\psi \tag{1.5}$$

and the geometric integral

$$\mathcal{J}_2 = \sum_{(1,2,3)}\gamma_1^2 = 1 \tag{1.6}$$

For this to be completely integrable one additional integral is missing, which, as is well known, exists in the Euler, Lagrange and Kovalevskaya cases for arbitrary values of the constant of the area integral and also in the Goryachev–Chaplygin case at the zero level of this integral.

2. THE LIMIT TRANSITION

We will now assume that the "length" of the body is much greater than its "width" and its "thickness", and that the body is prolate along its third axis. In order to formalize this, we will introduce a parameter $\varepsilon \neq 0$ such that

$$r_{ji} = \varepsilon(r'_{ji}+\varepsilon\rho_{ji}), \quad j = 1, 2 \tag{2.1}$$

We will assume that the parameter ε is sufficiently small and that the following relations are satisfied

$$\sum_{i\in\mathcal{J}} m_i r'_{ji} = 0, \quad j = 1, 2$$

Then

$$\Lambda_j = \varepsilon^2\Lambda'_j + \ldots, \quad M_j = \varepsilon^2 M'_j, \quad j = 1, 2; \quad \Lambda'_j = \sum_{i\in\mathcal{J}} m_i r'^2_{ji}, \quad M'_j = \sum_{i\in\mathcal{J}} m_i\rho_{ji}$$

Dropping the primes, we can represent Eqs (1.1) in the form

$$\begin{aligned}
(\varepsilon^2\Lambda_2+\Lambda_3+\ldots)\dot\omega_1 &= (\Lambda_3-\varepsilon^2\Lambda_2+\ldots)\omega_2\omega_3+\gamma_2M_3-\gamma_3\varepsilon^2M_2\\
(\Lambda_3+\varepsilon^2\Lambda_1+\ldots)\dot\omega_2 &= (\varepsilon^2\Lambda_1-\Lambda_3+\ldots)\omega_3\omega_1+\gamma_3\varepsilon^2M_1-\gamma_1M_3\\
\varepsilon^2(\Lambda_1+\Lambda_2+\ldots)\dot\omega_3 &= \varepsilon^2(\Lambda_2-\Lambda_1+\ldots)\omega_1\omega_2+\gamma_1\varepsilon^2M_2-\gamma_2\varepsilon^2M_1
\end{aligned} \tag{2.2}$$

Dividing the left- and right-hand sides in the last equation by ε^2 and then letting the parameter ε tend to zero, we have in the limit

$$\begin{aligned}
&\dot\omega_1 = \omega_1\omega_3+\mu_3\gamma_2, \quad \dot\omega_2 = -\omega_3\omega_1-\mu_3\gamma_1, \quad \dot\omega_3 = K\omega_1\omega_2+\gamma_1\mu_2-\gamma_2\mu_1\\
&K = (\Lambda_2-\Lambda_1)/(\Lambda_1+\Lambda_2), \quad \mu_j = M_j/(\Lambda_1+\Lambda_2), \quad j = 1, 2, \quad \mu_3 = M_3/L_3
\end{aligned} \tag{2.3}$$

Equations (2.3) must be supplemented by Poisson's equations (1.2). By taking appropriate limits in the first integrals $\mathcal{J}_0$ and $\mathcal{J}_1$, we can represent them in the form

$$\mathcal{J}_0 = \Lambda_3((\omega_1^2+\omega_2^2)/2+\mu_3\gamma_3) = h, \quad \mathcal{J}_1 = \Lambda_3(\omega_1\gamma_1+\omega_2\gamma_2) = p_\psi = \Lambda_3P \tag{2.4}$$

These equations were investigated in [1] for the case when $K = 0$ and $\mu_3 = 0$. Henceforth, without loss of generality, we will assume that $K \geq 0$.

Remarks. 1. If $K = 0$, the body is similar to a pencil, in which the "width" and "thickness" approximately coincide. The fact that μ_1 and μ_2 are non-zero denotes a slight asymmetry of the "point", while the quantity μ_3 corresponds to the longitudinal displacement of the centre of mass with respect to the suspension point. The case $K \neq 0$ denotes that a "student's ruler" is being considered, in which the "width" differs considerably from the "thickness".

2. It would be natural to assume that Eqs (2.3) possess the structure of the Poincaré–Chetayev equations. However, we know of no proof of this assumption.

3. SOME CASES OF THE EXISTENCE OF ADDITIONAL FIRST INTEGRALS

For Eqs (1.2) and (2.3) we can indicate some cases where additional first integrals exist.

The "Euler case". Suppose $\mu_1 = \mu_2 = \mu_3 = 0$. In this case Eqs (2.3) can be separated from Poisson's equations (1.2) and they can be considered independently of the latter. The additional integral can be represented in the form

$$\mathcal{J}_3 = (K\omega_2^2 + \omega_3^2)/2 = f \tag{3.1}$$

or in the form

$$\mathcal{J}_3 = (-K\omega_1^2 + \omega_3^2)/2 = g \tag{3.2}$$

In this case the equations of motion of the body turn out to be completely integrable. As in the classical Euler case, in the general case joint levels of the first integrals $\mathcal{J}_0$ and $\mathcal{J}_3$ are formed by a pair of curves, symmetrical about the origin of coordinates, each of which is a diffeomorph of a circle. In special cases, the joint levels consist either of a pair of symmetrical points or of a separatrice contour. This contour is situated at the zero level of integral (3.2). This special level of the first integral (3.2) is formed by a pair of intersecting planes

$$\mathcal{J}_\pm = \omega_3 \pm \sqrt{K}\omega_1 = 0 \tag{3.3}$$

It can be shown by a standard method, Routh's method, that when $K \neq 0$ in the case considered the set of steady motions, as in the Euler case, consists of uniform rotations about the axes of the body-fixed system of coordinates. In view of the assumption that K is non-negative, rotations about the first and third axes turn out to be stable, whereas rotation about the second axis is unstable.

In the general case, the equations of motion in the "Euler case" can be integrated in terms of elliptic functions. Some qualitative properties of the motion of such a system will be considered below.

None that, for dynamically symmetrical bodies, the equations of motion in the Euler case are obtained from the ones considered if we put $K = 0$ in them. In this case, the additional integral, as a rule, has the form

$$\mathcal{J}_3 = \omega_3$$

Remark. It can be shown by direct substitution of expressions (2.1) into the conditions for the existence of Kovalevskaya and Goryachev–Chaplygin integrals that these conditions fail the limit transition performed, and the corresponding additional integrals do not exist. The Lagrange case is not such – it requires an additional consideration.

The Hess case. As in the classical problem of the motion of a heavy rigid body about a fixed point, for the restricted formulation no splitting of the separatrice and the connected existence of linear particular integrals is observed. These integrals have the form (3.3) and they exist when the following conditions are satisfied

$$\lambda_1 = \mp\sqrt{K}\lambda_3, \quad \lambda_2 = 0 \tag{3.4}$$

respectively.

We will compare the particular integrals (3.3) with the Hess integral in the classical problem of the motion of a heavy rigid body about a fixed point. To do this we introduce the notation

$$I_1 = \Lambda_2 + \Lambda_3 \ (1, 2, 3)$$

and, to fix our ideas, we will assume $I_1 > I_2 > I_3$. Then, if

$$\mathbf{a} = (a_1, a_2, a_3)\colon a_1 = \sqrt{I_2^{-1} - I_1^{-1}}, \quad a_2 = 0, \quad a_3 = \sqrt{I_3^{-1} - I_2^{-1}}$$

the Hess integral has the form

$$F_\varepsilon = a_1 I_1 \omega_1 + \varepsilon^\circ a_3 I_3 \omega_3 = 0, \quad \varepsilon^\circ = \pm 1$$

where its existence is due to the fact that the following conditions are satisfied (compare with (3.4))

$$a_1 M_3 - \varepsilon^\circ a_3 M_1 = 0, \quad M_2 = 0$$

4. INTEGRATION OF THE EQUATIONS OF MOTION

In the special case when $K = 0$, a method of integrating the equations of motion (1.2) and (2.3) was proposed previously in [1] (see also [3, pp. 239–242]). The approach can also be used when the condition $K = 0$ is not satisfied.

We note first of all that the area integral $\mathcal{J}_1$ and one of Poisson's equations consist of a system of two algebraic equations

$$\omega_1\gamma_1 + \omega_2\gamma_2 = P, \quad \omega_2\gamma_1 - \omega_1\gamma_2 = \dot{\gamma}_3 \tag{4.1}$$

This system is linear in (ω_1, ω_2) and its solution has the form

$$\omega_1 = \frac{P\gamma_1 - \gamma_2\dot{\gamma}_3}{\gamma_1^2 + \gamma_2^2}, \quad \omega_2 = \frac{P\gamma_2 + \gamma_1\dot{\gamma}_3}{\gamma_1^2 + \gamma_2^2} \tag{4.2}$$

Substituting solution (4.2) into the energy integral $\mathcal{J}_0$, we have

$$\frac{1}{2}\frac{P^2 + \dot{\gamma}_3^2}{\gamma_1^2 + \gamma_2^2} + \mu_3\gamma_3 = H$$

Using the geometrical integral, this equation can be converted to the form

$$P^2 + \dot{\gamma}_3^2 = 2(1 - \gamma_3^2)(H - \mu_3\gamma_3) \tag{4.3}$$

which is closed with respect to γ_3. Recalling that

$$\gamma_1 = \sin\theta\sin\varphi, \quad \gamma_2 = \sin\theta\cos\varphi, \quad \gamma_3 = \cos\theta$$

where φ and θ are the angles of proper rotation and nutation, it can be shown that the equation describing the change in the angle of nutation can be separated from the equations for the two other angles describing the position of the system.

Equation (4.3) is identical with the equation describing the motion of a spherical pendulum after reducing the order according to Routh. When $\mu_3 \neq 0$ this equation can be integrated in elliptic functions. When $\mu_3 = 0$ it can be integrated in terms of elementary functions, where

$$\gamma_3 = A\cos[\omega(t + \alpha)], \quad \dot{\gamma}_3 = -A\omega\sin[\omega(t+\alpha)], \quad \omega = \sqrt{2H}, \quad A = \sqrt{1 - P^2/\omega^2} \tag{4.4}$$

The quantity ω plays the role of the frequency of the oscillations while A plays the role of their amplitudes.

We will introduce the variable ξ, such that

$$\omega_1 = \Omega\sin\xi, \quad \omega_2 = \Omega\cos\xi, \quad \Omega = \Omega(\gamma_3; \mu_3, H) = \sqrt{2(H - \mu_3\gamma_3)}, \quad \Omega(\gamma_3; 0, H) = \omega \tag{4.5}$$

and we will consider relations (4.1) as equations in (γ_1, γ_2). These equations are linear, and their general solution can be represented in the form

$$\gamma_1 = \frac{P\omega_1 + \dot{\gamma}_3\omega_2}{\omega_1^2 + \omega_2^2} = \frac{P\sin\xi + \dot{\gamma}_3\cos\xi}{\Omega} \tag{4.6}$$

$$\gamma_2 = \frac{P\omega_2 - \dot{\gamma}_3\omega_1}{\omega_1^2 + \omega_2^2} = \frac{P\cos\xi - \dot{\gamma}_3\sin\xi}{\Omega} \tag{4.7}$$

Now, differentiating the expression for ω_1 with respect to time, substituting the expression obtained and relation (4.7) into the first of equations (2.3) and reducing similar terms, we have

$$\dot{\xi} = \omega_3 + \mu_3 P/\Omega^2 \tag{4.8}$$

Differentiating the left- and right-hand sides of Eq. (4.8) with respect to time, substituting the expression for $\dot{\omega}_3$ from the last equation (2.3) into the right-hand side, and also replacing the quantities (ω_1, ω_2, γ_1, γ_2) by their values from relations (4.5)–(4.7), we obtain the second-order non-autonomous equation

$$\ddot{\xi} = K\Omega^2\sin\xi\cos\xi + \mu_2\frac{P\sin\xi + \dot{\gamma}_3\cos\xi}{\Omega} - \mu_1\frac{P\cos\xi - \dot{\gamma}_3\sin\xi}{\Omega} + \mu_3^2\frac{2P}{\Omega^4}\dot{\gamma}_3 \tag{4.9}$$

When $\mu_1 = \mu_2 = \mu_3 = 0$ this equation is completely integrable – its equivalence to the equations of motion of a mathematical pendulum can be proved using the replacement $\eta = 2\xi$. When $\mu_1 = \mu_2 = 0$ and for small values of the parameter μ_3, the non-integrability of Eq. (4.9) follows from the non-integrability of the equations of motion of a pendulum acted upon by a periodic torque.† We have thereby proved that there is no first integral in the case which could have been called the "Lagrange case". Finally, the non-integrability of Eq. (4.9) when $\mu_3 = 0$, $\mu_1\mu_2 \neq 0$ was proved in [4] by the method of splitting the separatrice.

Hence, when the non-degeneracy conditions are satisfied, the first integrals $\mathcal{J}_1$ and $\mathcal{J}_2$ define a three-dimensional invariant surface in $(\boldsymbol{\gamma}, \boldsymbol{\omega})$ space. The variables ξ, $\dot{\xi}$ and γ_3 can be regarded as coordinates on this surface. This indicates that, if the initial system is completely integrable, Eq. (4.9) is also completely integrable. The above results on the non-integrability of Eq. (4.9) prove the non-integrability of the equations of motion of a heavy rigid body in the "restricted" formulation.

We draw attention to the fact that chaotic motion, related to the non-integrability, develops with respect to the angle of rotation, whereas the dynamics with respect to the angles of precession and nutation remain regular. This is the basis for dividing the motions with respect to the angles of nutation and precession, on the one hand, and with respect to the proper rotation on the other, which is precisely the restricted formulation of the problem considered.

Note also that a similar separation of the motions is also possible in a number of other classical problems on the motion of a rigid body, in particular, in the problem of the motion of a body in an unbounded volume of an ideal incompressible fluid.

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